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Upper Bounds on T_c for One-Dimensional Ising Systems

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We present upper bounds on the critical temperature of one-dimensional Ising models with long-range, $1/n^{\alpha}$ interactions, where $1 < \alpha \leqslant 2$. In particular for the often studied case of $\alpha = 2$ we have an upper bound on T_c which is less than the T_c found by a number of approximation techniques. Also for the case where α is small, such as $\alpha = 1.1$, we obtain rigorous bounds which are extremely close, within 1.0%, to those found by approximation methods.

KEY WORDS: Ising model; long-range interactions; critical temperature.

1. INTRODUCTION

For some time it has been known that ferromagnetic, one-dimensional Ising spin systems have a phase transition if the interaction strength falls off slowly enough. In particular if we consider Ising spins, $s = \pm 1$, given by the Hamiltonian

$$\mathscr{H} = -\sum_{i<j} \frac{J}{|i-j|^{\alpha}} s_i s_j - \sum_i h_i s_i \tag{1}$$

where $J > 0$, $h_i \geqslant 0$, and $|i-j|$ is the distance between the ith and the jth sites with the distance between nearest neighbor sites set equal to one, then Dyson[1,2] proved there exists a phase transition for $1 < \alpha < 2$. For $\alpha > 2$ there is no phase transition.[3] Thus $\alpha = 2$ is the critical power and a delicate case for analysis. Frohlich and Spencer[4] gave the first proof of the existence of a phase transition for this case. More recently Aizenman *et al.*[5] and Imbrie and Newman[6] have extended and sharpened many

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results for the $\alpha = 2$ case. Of particular interest for this paper is the following dichotomy proven in ref. 5:

$$\text{either} \quad M = 0 \quad \text{or} \quad \beta_c J M^2 \geqslant 1/2 \tag{2}$$

where M is the spontaneous magnetization for the system and $\beta = 1/kT$. The actual statement of ref. 5 considers q-state Potts models and hence is more general; however, we consider only the Ising ($q = 2$) case.

In this paper we establish rigorous lower bounds on β_c for the above model. Particular attention is paid to the case $\alpha = 2$ and to the opposite end of the α range, i.e., when α is in the range $1.0 < \alpha \leqslant 1.2$. In the latter case we can use a method due to Vigfusson[7] or one due to the present author[8] to obtain the bounds. Both have been used previously for only the $\alpha = 2$ case. However, both can be shown to be very effective when applied to those systems where the interaction falls off very slowly. For the case of $\alpha = 2$ we use the method of Vigfusson along with the dichotomy presented in (2) to improve significantly upon previous bounds. In both cases we compare our bounds not only to previous bounds, but also to a number of results based on a variety of approximation techniques and in both cases our rigorous bounds are strong enough to rule out some of the approximation results. One motivation for presentation of these results is to establish some good rigorous bounds to which approximation results can be compared.

Besides the dichotomy presented above we need the following theorem of Vigfusson.

Theorem. If, in (1), a pair coupling is replaced by a coupling of the individual spins to local external fields m_i and m_j through a term $(J/|i-j|^{\alpha})(s_i m_j + s_j m_i)$, and the m_i and m_j are chosen such that they satisfy $m_i \geqslant \langle s_i \rangle$ and $m_j \geqslant \langle s_j \rangle$ (as calculated in the new system), then for any k, $\langle s_k \rangle$ in the new system is larger than or equal to its value in the original system.

The brackets around s_i, s_j, and s_k in the theorem denote the usual thermal average. The statement of the theorem can be made more general than the above statement but this is unnecessary for the present application. For both the more general statement and the proof see Vigfusson.[9]

2.1. $0 < \alpha \leqslant 1.2$

We begin with the case of α small and present only the Vigfusson method. For α small the method presented in ref. 8 basically duplicates the bounds achieved using Vigfusson's method and requires about the same

amount of computation to achieve the bounds. Therefore these results will not be presented.

To make use of the theorem presented in the introduction one needs to substitute for enough of the pair interactions in (1) to result in a system where one can readily compute the $\langle s_i \rangle$ and check that $m_i \geqslant \langle s_i \rangle$ for all i. Then one needs to find the greatest βJ such that $m_i > \langle s_i \rangle$ remains valid as $m_i \to 0$, thereby assuring that $\langle s_i \rangle \to 0$. Vigfusson presents one approach in ref. 7. There he substitutes for the pair interactions in such a way that the system is reduced to a chain of spins with the structure shown in Fig. 1a. The structure of the chain is such that any $\langle s_i \rangle$ can be calculated by knowing the eigenvalues of a 2×2 transfer matrix (for more details see ref. 7). For a small unit cell, such as the four-site unit cell shown in Fig. 1a, the elements of the transfer matrix can be obtained by hand. For larger cells one can make use of one of the symbolic manipulation programs available. Vigfusson for the $\alpha = 2$ case used unit cells containing up to 20 sites. For our calculations with $1.0 < \alpha \leqslant 1.2$ we have only gone to unit cells containing 12 sites. One will see that for α in the range we are considering, by the time one considers 12-site unit cells one has very good bounds and going to larger cells will not improve them significantly.

We present results based on two methods in handling the fields m_i. In one case we consider only a single mean field m and then require $m > \langle s_i \rangle$ for all i. This simplifies the calculations and was the procedure used by Vigfusson for his $1/n^2$ results. For our other results we use the full proce-

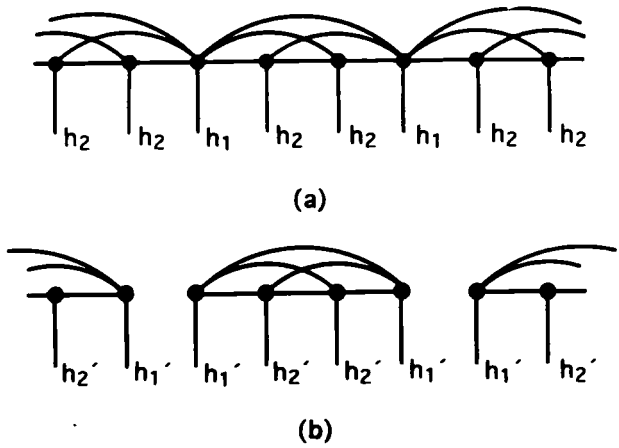

Fig. 1. (a) Replacement of pair interactions to the point where one has four-site unit cells. Local magnetic fields h_1 and h_2 replace the deleted pair interactions. (b) Replacement of pair interactions to the point where one has independent four-site clusters. Here local magnetic fields h_1' and h_2' replace the deleted pair interactions.

Table I. Bounds and Approximations for $\beta_c J$ for $\alpha = 1.1$ and 1.2

	Bounds			
	$\alpha = 1.1$		$\alpha = 1.2$	
Unit cell	Procedure 1	Procedure 2	Procedure 1	Procedure 2
4-site	$\beta_c J \geq 0.04748$	$\beta_c J \geq 0.04749$	$\beta_c J \geq 0.09110$	$\beta_c J \geq 0.09117$
8-site	$\beta_c J \geq 0.04751$	$\beta_c J \geq 0.04752$	$\beta_c J \geq 0.09136$	$\beta_c J \geq 0.09147$
12-site	$\beta_c J \geq 0.04752$	$\beta_c J \geq 0.04753$	$\beta_c J \geq 0.09144$	$\beta_c J \geq 0.09162$

	Approximations	
	$\alpha = 1.1$	$\alpha = 1.2$
Bethe lattice[11]	$\beta_c J \approx 0.0477$	$\beta_c J \approx 0.0927$
Series expansions[12]	—	$\beta_c J \approx 0.0919$
Cluster mean-field[13]	$\beta_c J \approx 0.0478$	$\beta_c J \approx 0.0934$
Coherent anomaly method[14]	$\beta_c J \approx 0.0473$	$\beta_c J \approx 0.0927$
Finite-range scaling[15]	$\beta_c J \approx 0.0505$	$\beta_c J \approx 0.0923$

dure with differing m_i. Naturally this results in better bounds, but at the cost of much more complicated calculations. We present both results in Table I so that one can see the amount of improvement one obtains using the full procedure. Along with our rigorous bounds we give the results of five different approximations for the critical βJ. The results based on 4-, 8-, and 12-site unit cells are given. One sees that even the 4-site unit cell gives relatively good bounds in this region of small α. One also sees that the rigorous bounds are very close to the values found by approximation. The method becomes less effective as α increases.

3. $\alpha = 2.0$

The case of $\alpha = 2.0$ has been by far the most studied case. Vigfusson, using the theorem in the introduction and the transfer matrix method discussed in the previous section, produced the best bound in a long series of rigorous bounds, establishing $\beta_c J > 0.441$. With the results of Aizenman *et al.*[5] one has immediately that $\beta_c J \geq 0.500$ from (2). Vigfusson required that as $\beta \to \beta_c$ the spontaneous magnetization $M \to 0$, but from the dichotomy given in (2) one need no longer require that $M \to 0$ as $\beta \to \beta_c$, but rather require that $\beta_c J M^2 \geq 1/2$. In other words, one wants to find the greatest β such that $\beta J M^2 \geq 1/2$ while still requiring $m_i \geq \langle s_i \rangle$ for all i. This allows for a significant improvement in the bounds one can obtain

Table II. Bounds and Approximations for $\beta_c J$ for $\alpha = 2.0$

	Bounds	Approximations	
	Transfer matrix		
4-site	$\beta_c J \geq 0.59229$	Bethe lattice[12]	$\beta_c J \approx 0.658$
8-site	$\beta_c J \geq 0.60004$	Matvienko[15]	$\beta_c J \approx 0.657$
12-site	$\beta_c J \geq 0.60504$	Monte Carlo[16]	$\beta_c J \approx 0.630$
16-site	—	Renormalization group[17]	$\beta_c J \approx 0.615$
20-site	—	Series Expansion[11]	$\beta_c J \approx 0.612$
24-site	—	Finite-range scaling[14]	$\beta_c J \approx 0.596$
	Cluster	ζ-Function[18]	$\beta_c J \approx 0.590$
4-site	$\beta_c J \geq 0.58517$	Coherent anomaly[13]	$\beta_c J \approx 0.571$
8-site	$\beta_c J \geq 0.59643$	Variational method[19]	$\beta_c J \approx 0.500$
12-site	$\beta_c J \geq 0.60237$	Mean-field	$\beta_c J \approx 0.313$
16-site	$\beta_c J \geq 0.60626$		
20-site	$\beta_c J \geq 0.60909$		
24-site	$\beta_c J \geq 0.61128$		

using the theorem given in the introduction. Again using the transfer matrix approach of the previous section, results for the 4-, 8-, and 12-site unit cells are given in Table II. In addition to these results we present the bounds based on what we denote as the cluster method. Here we replace the pair interactions to a point where we have a collection of independent clusters each with n sites; see Fig. 1b. In this procedure we do not need to deal with a transfer matrix and most importantly taking derivatives of the matrix elements in order to calculate the necessary thermal averages. This has allowed us to go to clusters containing up to 24 sites. The results one obtains using this method are also presented in Table II. Finally Table II also presents the approximate value of $\beta_c J$ from a large variety of methods. It is worth noting that our improvement in bounds has resulted in a situation where approximately half of the approximations are seen to give values for $\beta_c J$ which our bounds show to be too low.

4. CONCLUSION

We see from the above that a straightforward use of the method of Vigfusson, previously used only for the $\alpha = 2.0$ case, results in what are very good lower bounds on $\beta_c J$ for the case where the interaction falls off very slowly. In addition, for the intensely studied $\alpha = 2.0$ if Vigfusson's method is combined with the dichotomy of (2) one can likewise achieve very good lower bounds on $\beta_c J$. In both cases the bounds can be used as a test for the previously employed approximation methods to obtain $\beta_c J$.

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